



## Introduction to AntiGroups

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◊ In commemoration of the 60th birthday of the author

### Abstract

The notion of AntiGroups is formally presented in this paper. A particular class of AntiGroups of type-AG[4] is studied with several examples and basic properties presented. In AntiGroups of type-AG[4], the existence of an inverse is taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity are taking to be either partially true, partially indeterminate or partially false for some elements. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that distinct left(right) cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4].

**Keywords:** NeutroGroup, AntiGroup, AntiSubgroup, AntiQuotientGroup, AntiGroupHomomorphism.

### 1 Introduction

Neutrosophic logic (NL) introduced by Smarandache in 1995 is an alternative to the existing classical logics and the generalization of fuzzy logic (FL) of Zadeh [12] and intuitionistic fuzzy logic (IFL) of Atanassov [5]. Neutrosophic logic is a non-classical logic that can be used as a mathematical tool to model situations characterized by uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. In neutrosophic logic [11], each proposition is estimated to have percentage of truth in a subset  $T$ , the percentage of indeterminacy in a subset  $I$ , and the percentage of falsity in a subset  $F$  where  $(T, I, F)$  are standard or non-standard subsets of the non-standard interval  $]^{-}0, 1^{+}[$ , where  $n_{\inf} = \inf T + \inf I + \inf F \geq^{-} 0$ , and  $n_{\sup} = \sup T + \sup I + \sup F \leq 3^{+}$ . Statically,  $(T, I, F)$  are subsets but dynamically, they are functions/operators depending on many known or unknown parameters. In NL, if  $\langle A \rangle$  is an idea, or proposition, theory, law, axiom, event, concept, entity etc., there correspond  $\langle \text{Non-}A \rangle = \langle \text{Anti-}A \rangle$  which is the opposite of  $\langle A \rangle$  and  $\langle \text{Neut-}A \rangle$  which stands for what is neither  $\langle A \rangle$  nor  $\langle \text{Anti-}A \rangle$ , that is neutrality in between the two extremes. Consequently in NL, it is possible to have the triad ( $\langle A \rangle, \langle \text{Neut-}A \rangle, \langle \text{Anti-}A \rangle$ ). The non-restriction in NL allows for paraconsistent, dialetheist, and incomplete information to be characterized. This special and unique feature of NL has made it applicable in solving problems involving uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction etc. arising from science, social science, engineering, technology, computer science, artificial intelligence, ICT, robotics etc.

The indeterminacy factor  $I$  in NL is fundamental in the formulation and establishment of any neutrosophic algebraic structure. Given any classical algebraic structure  $(X, *)$ , a structure  $X(I) = \langle X \cup I \rangle$  generated by  $X$  and  $I$  under the binary operation  $*$  of  $X$  is called a neutrosophic algebraic structure with its name derived from the name of  $X$ . For instance, if  $X$  is a group, then  $X(I)$  is called a neutrosophic group. Since  $I^{-1}$ , the inverse of  $I$  does not exist, finding  $x^{-1}$ , the inverse of any neutrosophic element  $x \in X(I)$  becomes difficult and impossible. Consequently, algebraic manipulations of the elements of  $X(I)$  become restrictive. The recent introduction of the concepts of NeutroStructures and AntiStructures have lessened the restrictiveness of the algebraic manipulations of the elements of neutrosophic algebraic structures imposed by the neutrosophic element  $I$ .

In any classical algebraic structure  $(X, *)$ , the law of composition of the elements with respect to the binary operation  $*$  is well defined for all the elements of  $X$  that is,  $x * y \in X \quad \forall x, y \in X$ . All the axioms like associativity, commutativity, distributivity, monotonicity etc. defined on  $X$  with respect to  $*$  are totally true for all the elements of  $X$ . The compositions of elements of  $X$  this way are restrictive and do not reflect the reality. They do not give rooms for compositions that are either partially defined, partially undefined (indeterminate), and partially outerdefined or totally outerdefined with respect to  $*$ . However in the domain of knowledge, science and reality, the law of composition and axioms defined on  $X$  may either be only partially defined (partially true), or partially undefined (partially false), or totally undefined (totally false) with respect to the binary operation  $*$ . In 2019, Smarandache [10] addressed the problem of allowing the law of composition on  $X$  to be either partially defined and partially undefined or totally undefined by introducing the notions of NeutroDefined and AntiDefined laws, as well as the notions of NeutroAxiom and AntiAxiom inspired by NL he introduced in 1995. The work of Smarandache in [10] has given birth to the new fields of research called NeutroStructures and AntiStructures. For any classical algebraic law or axiom defined on  $X$ , there correspond neutrosophic triads ( $< \text{Law} >$ ,  $< \text{NeutroLaw} >$ ,  $< \text{AntiLaw} >$ ) and ( $< \text{Axiom} >$ ,  $< \text{NeutroAxiom} >$ ,  $< \text{AntiAxiom} >$ ) respectively. In [9], Smarandache studied NeutroAlgebras and AntiAlgebras and in [8], he studied Partial Algebras, Universal Algebras, Effect Algebras and Boole's Partial Algebras and he showed that NeutroAlgebras are generalization of Partial Algebras. In [7], Rezaei and Smarandache studied Neutro-BE-algebras and Anti-BE-algebras and they showed that any classical algebra  $S$  with  $n$  operations (laws and axioms) where  $n \geq 1$  will have  $(2^n - 1)$  NeutroAlgebras and  $(3^n - 2^n)$  AntiAlgebras. In [2], Agboola et al. studied NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems  $\mathbb{N}$ ,  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$  and  $\mathbb{C}$  and in [3], Agboola studied NeutroGroups by considering three NeutroAxioms (NeutroAssociativity, existence of NeutroNeutral element and existence of NeutroInverse element). In addition, he studied NeutroSubgroups, NeutroCyclicGroups, NeutroQuotientGroups and NeutroGroupHomomorphisms. He showed that generally, Lagrange's theorem and fundamental homomorphism theorem of the classical groups do not hold in the class of NeutroGroups studied. In [4], Agboola introduced and studied NeutroRings by considering three NeutroAxioms (NeutroAbelianGroup (additive), NeutroSemigroup (multiplicative) and NeutroDistributivity (multiplication over addition)). He presented Several results and examples on NeutroRings, NeutroSubrings, NeutroIdeals, NeutroQuotientRings and NeutroRingHomomorphisms. He showed that that the fundamental homomorphism of the classical rings holds in the class of NeutroRings considered. Motivated and inspired by the work of Rezaei and Smarandache in [7], Agboola in [1] revisited the NeutroGroups by studying a particular class of NeutroGroups and presented their basic and elementary properties. In the present paper however, the concept of AntiGroups is formally presented. A particular class of AntiGroups is studied with presentation of several examples and basic properties. It is shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups studied. Specifically, it is shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it is shown that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups studied in this paper.

## 2 Preliminaries

In this section, we will give some definitions and results that will be used later in the paper.

**Definition 2.1.** [8]

- (i) A classical operation is an operation well defined for all the set's elements.
- (ii) A NeutroOperation is an operation partially well defined, partially indeterminate, and partially outer defined on the given set.
- (iii) An AntiOperation is an operation that is outer defined for all set's elements.
- (iv) A classical law/axiom defined on a nonempty set is a law/axiom that is totally true (i.e. true for all set's elements).
- (v) A NeutroLaw/NeutroAxiom (or Neutrosophic Law/Neutrosophic Axiom) defined on a nonempty set is a law/axiom that is true for some set's elements [degree of truth (T)], indeterminate for other set's elements [degree of indeterminacy (I)], or false for the other set's elements [degree of falsehood (F)], where  $T, I, F \in [0, 1]$ , with  $(T, I, F) \neq (1, 0, 0)$  that represents the classical axiom, and  $(T, I, F) \neq (0, 0, 1)$  that represents the AntiAxiom.

- (vi) An AntiLaw/AntiAxiom defined on a nonempty set is a law/axiom that is false for all set's elements.
- (vii) A NeutroAlgebra is an algebra that has at least one NeutroOperation or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for other elements), and no Anti-Operation or AntiAxiom.
- (viii) An AntiAlgebra is an algebra endowed with at least one AntiOperation or at least one AntiAxiom.

**Theorem 2.2.** [<sup>7</sup>] Let  $\mathbb{U}$  be a nonempty finite or infinite universe of discourse and let  $S$  be a finite or infinite subset of  $\mathbb{U}$ . If  $n$  classical operations (laws and axioms) are defined on  $S$  where  $n \geq 1$ , then there will be  $(2^n - 1)$  NeutroAlgebras and  $(3^n - 2^n)$  AntiAlgebras.

**Definition 2.3.** [Classical group][<sup>6</sup>]

Let  $G$  be a nonempty set and let  $* : G \times G \rightarrow G$  be a binary operation on  $G$ . The couple  $(G, *)$  is called a classical group if the following conditions hold:

- (G1)  $x * y \in G \forall x, y \in G$  [closure law].
  - (G2)  $x * (y * z) = (x * y) * z \forall x, y, z \in G$  [axiom of associativity].
  - (G3) There exists  $e \in G$  such that  $x * e = e * x = x \forall x \in G$  [axiom of existence of neutral element].
  - (G4) There exists  $y \in G$  such that  $x * y = y * x = e \forall x \in G$  [axiom of existence of inverse element] where  $e$  is the neutral element of  $G$ .
- If in addition  $\forall x, y \in G$ , we have
- (G5)  $x * y = y * x$ , then  $(G, *)$  is called an abelian group.

**Definition 2.4.** [AntiSophification of the law and axioms of the classical group][<sup>1</sup>]

- (AG1) For all the duplets  $(x, y) \in G$ ,  $x * y \notin G$  [AntiClosureLaw].
- (AG2) For all the triplets  $(x, y, z) \in G$ ,  $x * (y * z) \neq (x * y) * z$  [AntiAxiom of associativity (AntiAssociativity)].
- (AG3) There does not exist an element  $e \in G$  such that  $x * e = e * x = x \forall x \in G$  [AntiAxiom of existence of neutral element (AntiNeutralElement)].
- (AG4) There does not exist  $u \in G$  such that  $x * u = u * x = e \forall x \in G$  [AntiAxiom of existence of inverse element (AntiInverseElement)] where  $e$  is an AntiNeutralElement in  $G$ .
- (AG5) For all the duplets  $(x, y) \in G$ ,  $x * y \neq y * x$  [AntiAxiom of commutativity (AntiCommutativity)].

**Definition 2.5.** [AntiGroup][<sup>1</sup>]

An AntiGroup  $AG$  is an alternative to the classical group  $G$  that has at least one AntiLaw or at least one of  $\{AG1, AG2, AG3, AG4\}$ .

**Definition 2.6.** [AntiAbelianGroup][<sup>1</sup>]

An AntiAbelianGroup  $AG$  is an alternative to the classical abelian group  $G$  that has at least one AntiLaw or at least one of  $\{AG1, AG2, AG3, AG4\}$  and  $AG5$ .

**Theorem 2.7.** [<sup>1</sup>] Let  $(G, *)$  be a finite or infinite classical group. Then there are 65 types of AntiGroups.

**Theorem 2.8.** [<sup>1</sup>] Let  $(G, *)$  be a finite or infinite classical abelian group. Then there are 211 types of AntiAbelianGroups.

**Definition 2.9.** Let  $(AG, *)$  be an AntiGroup.  $AG$  is said to be finite of order  $n$  if the cardinality of  $AG$  is  $n$  that is  $o(AG) = n$ . Otherwise,  $AG$  is called an infinite AntiGroup and we write  $o(AG) = \infty$ .

Since there are many types of AntiGroups, in what follows, AntiGroups will be classified and named type- $AG[.]$  according to which of  $AG1-AG5$  is(are) satisfied. If only  $AG1$  is satisfied, the AntiGroup will be called of type- $AG[1]$ , type- $AG[3,4]$  if only  $AG3$  and  $AG4$  are satisfied and so on. AntiGroups of type- $AG[1,2,3,4]$  or of type- $AG[1,2,3,4,5]$  will be called trivial AntiGroups or trivial AntiAbelianGroups respectively.

**Example 2.10.** Let  $AG = \mathbb{Q}_+^*$  be the set of all irrational positive numbers and consider algebraic structure  $(AG, .)$  where “.” is the ordinary multiplication of real numbers. Then  $(AG, .)$  is an infinite trivial AntiGroup.

**Example 2.11.** Let  $AG = \mathbb{N}$ .

- (i) Let  $*$  be a binary operation on  $AG$  defined  $\forall x, y \in AG$  by

$$x * y = x + y + xy.$$

Then  $(AG, *)$  is a finite AntiGroup of type-AG[3,4].

- (ii) Let  $*$  be a binary operation on  $AG$  defined  $\forall x, y \in AG$  by

$$x * y = x + y + 1.$$

Then  $(AG, *)$  is a finite AntiGroup of type-AG[3,4].

### 3 A Study of Finite AntiGroups of Type-AG[4]

In this section, we are going to study a particular class of AntiGroups  $(AG, *)$  where  $G4$  is totally false for all the elements of  $AG$  while  $G1, G2, G3$  and  $G5$  are either partially true, partially indeterminate or partially false for some elements of  $AG$ .

**Definition 3.1.** Let  $(AG, *)$  and  $(AH, \circ)$  be AntiGroups of type-AG[4]. The direct product of  $AG$  and  $AH$  denoted by  $AG \times AH$  is defined by

$$AG \times AH = \{(g, h) : g \in AG, h \in AH\}.$$

**Proposition 3.2.** Let  $(AG, *)$  and  $(AH, \circ)$  be AntiGroups of type-AG[4] and let  $\otimes$  be a binary operation on  $AG \times AH$  defined by

$$(g, h) \otimes (x, y) = (g * x, h \circ y) \quad \forall (g, h), (x, y) \in AG \times AH.$$

Then  $(AG \times AH, \otimes)$  is an AntiGroup of type-AG[4].

*Proof.* The proof follows from the definition of AntiGroups of type-AG[4] and the definition of direct product of AntiGroups of type-AG[4].  $\square$

**Proposition 3.3.** Let  $(AG, *)$  be an AntiGroup of type-AG[4] and let  $g, x, y \in AG$ . Then

- (i)  $g * x = g * y \not\Rightarrow x = y$ .
- (ii)  $x * g = y * g \not\Rightarrow x = y$ .

*Proof.* Since  $g^{-1}$  does not exist and  $*$  is NeutroAssociative, the required results follow.  $\square$

**Proposition 3.4.** Let  $(AG, *)$  be an AntiGroup of type-AG[4],  $x, y \in AG$  and let  $m, n \in \mathbb{N}$ . Then

- (i)  $x^{m+1} \neq x^m * x$ .
- (ii)  $x^{-m} \neq (x^{-1})^m$ .
- (iii)  $x^m * x^{-m} \neq N_e$  where  $N_e$  is a NeutroNeutralElement in  $AG$ .
- (iv)  $x^m * x^n \neq x^{m+n}$ .
- (v)  $(x^m)^n \neq x^{mn}$ .
- (vi)  $(x * y)^m \neq x^m * y^m$ .

*Proof.* Since  $x^{-1}$  does not exist and  $*$  is NeutroAssociative, the required results follow.  $\square$

**Corollary 3.5.** An AntiGroup  $(AG, *)$  of type-AG[4] cannot be generated by an element  $x \in AG$  and hence cannot be cyclic.

**Definition 3.6.** Let  $(AG, *)$  be an AntiGroup of type-AG[4]. A nonempty subset  $AH$  of  $AG$  is called an AntiSubgroup of  $AG$  if  $(AH, *)$  is also an AntiGroup of the same type as  $AG$ . Otherwise, if  $(AH, *)$  is an AntiGroup of a type different from the type of  $AG$ , then  $AH$  is called a QuasiAntiSubgroup of  $AG$ .

**Definition 3.7.** Let  $(AG, *)$  be an AntiGroup of type-AG[4] and let  $AH$  and  $AK$  be AntiSubgroups of  $AG$ . The set  $A * B$  is defined by

$$A * B = \{x \in AG : x = h * k \text{ for some } h \in AH, k \in AK\}.$$

**Proposition 3.8.** Let  $AH$ ,  $AK$  and  $AL$  be AntiSubgroups of an AntiGroup  $(AG, *)$  of type-AG[4]. Then

- (i)  $AH * AH \neq AH$ .
- (ii)  $AH * AK \neq AK * AH$ .
- (iii)  $AH * (AK * AL) \neq (AH * AK) * AL$ .

*Proof.* Obvious. □

**Definition 3.9.** Let  $(AG, *)$  be an AntiGroup of type-AG[4] and let  $a \in AG$  be a fixed element.

- (i) An AntiCenter of  $AG$  denoted by  $AZ(AG)$  is a set defined by

$$AZ(AG) = \{x \in AG : x * g \neq g * x \quad \forall g \in AG\}.$$

- (ii) An AntiCentralizer of  $a \in AG$  denoted by  $AC_a$  is a set defined by

$$AC_a = \{g \in AG : g * a \neq a * g\}.$$

**Example 3.10.** Let  $\mathbb{U} = \{a, b, c, d, e, f\}$  be a universe of discourse and let  $AG = \{a, b, c, e\}$  be a subset of  $\mathbb{U}$ .

- (i) Let  $*$  be a binary operation defined on  $AG$  as shown in the Cayley table below.

*	a	b	c	e
a	d	a	c	e
b	a	d	e	c
c	b	a	f	e
e	a	b	c	f

It is evident from the table that  $G1, G2, G3, G5$  are either partially true or partially false with respect to  $*$  but  $G4$  is totally false for all the elements of  $AG$ . Hence  $(AG, *)$  is a finite AntiGroup of type-AG[4].

- (ii) Let  $*$  be a binary operation defined on  $AG$  as shown in the Cayley table below.

*	e	a	b	c
e	d	a	b	c
a	a	f	c	b
b	b	a	?	c
c	c	b	a	?

It is evident from the table that  $G1, G2, G3, G5$  are either partially true, partially indeterminate or partially false with respect to  $*$  but  $G4$  is totally false for all the elements of  $AG$ . Hence  $(AG, *)$  is a finite AntiGroup of type-AG[4].

**Example 3.11.** Let  $(AG, *)$  be the AntiGroup of Example 3.10(i) and let  $AH_1 = \{a, b, e\}$  and  $AH_2 = \{b, c, e\}$  be two subsets of  $AG$ . Let  $*$  be defined on  $AH_1$  and  $AH_2$  as shown in the Cayley tables below:

$AH_1 :$	<table border="1" style="border-collapse: collapse; width: 100%; text-align: center;"> <tr> <td>*</td><td>a</td><td>b</td><td>e</td></tr> <tr> <td>a</td><td>d</td><td>a</td><td>e</td></tr> <tr> <td>b</td><td>a</td><td>d</td><td>c</td></tr> <tr> <td>e</td><td>a</td><td>b</td><td>f</td></tr> </table>	*	a	b	e	a	d	a	e	b	a	d	c	e	a	b	f	$AH_2 :$	<table border="1" style="border-collapse: collapse; width: 100%; text-align: center;"> <tr> <td>*</td><td>b</td><td>c</td><td>e</td></tr> <tr> <td>b</td><td>d</td><td>e</td><td>c</td></tr> <tr> <td>c</td><td>a</td><td>f</td><td>e</td></tr> <tr> <td>e</td><td>b</td><td>c</td><td>f</td></tr> </table>	*	b	c	e	b	d	e	c	c	a	f	e	e	b	c	f
*	a	b	e																																
a	d	a	e																																
b	a	d	c																																
e	a	b	f																																
*	b	c	e																																
b	d	e	c																																
c	a	f	e																																
e	b	c	f																																

It can easily be seen from the tables that  $AH_1$  is an AntiSubgroup of  $AG$  while  $AH_2$  is a QuasiAntiSubgroup of  $AG$ . It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{a, b, c, e\} = AG, \\ NH_1 \cap NH_2 &= \{b, e\}, \end{aligned}$$

from which it is deduced that  $NH_1 \cup NH_2$  is an AntiSubgroup of  $AG$  but  $AH_1 \cap AH_2$  is a QuasiAntiSubgroup of  $AG$  as it is evident in the Cayley table below:

$NH_1 \cap NH_2 :$									
<table border="1"> <tr> <td>*</td> <td>b</td> <td>e</td> </tr> <tr> <td>b</td> <td>d</td> <td>c</td> </tr> <tr> <td>e</td> <td>b</td> <td>f</td> </tr> </table>	*	b	e	b	d	c	e	b	f
*	b	e							
b	d	c							
e	b	f							

$AZ(AG) = \{a, b, c, e\} = AG$  is an AntiSubgroup of  $AG$ . Also,  $AC_a = AC_b = AC_c = AC_e = \{a, b, c, e\} = AG$  are AntiSubgroups of  $AG$ .

**Example 3.12.** Let  $(AG, *)$  be the AntiGroup of Example 3.10(ii) and let  $AH_1 = \{e, a, b\}$  and  $AH_2 = \{e, b, c\}$  be two subsets of  $AG$ . Let  $*$  be defined on  $AH_1$  and  $AH_2$  as shown in the Cayley tables below:

$AH_1 :$																
<table border="1"> <tr> <td>*</td> <td>e</td> <td>a</td> <td>b</td> </tr> <tr> <td>e</td> <td>d</td> <td>a</td> <td>b</td> </tr> <tr> <td>a</td> <td>a</td> <td>f</td> <td>c</td> </tr> <tr> <td>b</td> <td>b</td> <td>a</td> <td>?</td> </tr> </table>	*	e	a	b	e	d	a	b	a	a	f	c	b	b	a	?
*	e	a	b													
e	d	a	b													
a	a	f	c													
b	b	a	?													

  

$AH_2 :$																
<table border="1"> <tr> <td>*</td> <td>e</td> <td>b</td> <td>c</td> </tr> <tr> <td>e</td> <td>d</td> <td>b</td> <td>c</td> </tr> <tr> <td>b</td> <td>b</td> <td>?</td> <td>c</td> </tr> <tr> <td>c</td> <td>c</td> <td>a</td> <td>?</td> </tr> </table>	*	e	b	c	e	d	b	c	b	b	?	c	c	c	a	?
*	e	b	c													
e	d	b	c													
b	b	?	c													
c	c	a	?													

It can easily be seen from the tables that  $AH_1$  and  $AH_2$  are AntiSubgroups of  $AG$ . It is noted that Lagranges' theorem does not hold. It is also noted that:

$$\begin{aligned} AH_1 \cup AH_2 &= \{e, a, b, c\} = AG, \\ NH_1 \cap NH_2 &= \{e, b\}, \end{aligned}$$

from which it is deduced that  $NH_1 \cup NH_2$  is an AntiSubgroup of  $AG$  and  $AH_1 \cap AH_2$  is an AntiSubgroup of  $AG$  as it is evident in the Cayley table below:

$NH_1 \cap NH_2 :$									
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*	e	b							
e	d	f							
b	b	?							

$AZ(AG) = \{a, b, c\}$  is a QuasiAntiSubgroup of  $AG$ . Also,  $AC_a = AC_b = \{a, b, c\}$ ,  $AC_c = \{b, c\}$  are QuasiAntiSubgroups of  $AG$  and  $AC_e = \{\} = \emptyset$  is neither an AntiSubgroup nor a QuasiAntiSubgroup of  $AG$ .

**Example 3.13.** (i) Let  $AG = \mathbb{Z}_4 = \{0, 1, 2, 3\}$  and let  $\oplus$  be a binary operation on  $AG$  as defined in the Cayley table.

$\oplus$	0	1	2	3
0	0 or 1	1	2	3
1	1	2	3	0 or 2
2	2	3	0 or 3	1
3	3	?	1	2

Then  $(AG, \oplus)$  is a finite AntiGroup of type-AG[4] and  $AH = \{0, 1, 2\}$  is an AntiSubgroup of  $AG$ .

(ii) Let  $AG = \{1, 2, 3, 4\} \subseteq \mathbb{Z}_5$  and let  $\otimes$  be a binary operation on  $AG$  as defined in the Cayley table.

$\otimes$	1	2	3	4
1	?	2	3	4
2	2	4	0	3
3	3	?	4	2
4	4	3	2	0

Then  $(AG, \otimes)$  is a finite AntiGroup of type-AG[4] and  $AH = \{1, 2, 3\}$  is an AntiSubgroup of  $AG$ .

**Definition 3.14.** Let  $AH$  be an AntiSubgroup of the AntiGroup  $(AG, *)$  of type-AG[4] and let  $x \in AG$ .

(i)  $x * AH$  the left coset of  $AH$  in  $AG$  is defined by

$$x * AH = \{x * h : h \in AH\}.$$

(ii)  $AH * x$  the right coset of  $AH$  in  $AG$  is defined by

$$AH * x = \{h * x : h \in AH\}.$$

(iii)  $AH$  is called a normal AntiSubgroup if  $x * AH = AH * x$  for at least one  $x \in AG$ .

(iv) The number of distinct left or right cosets of  $AH$  in  $AG$  is called the index of  $AH$  in  $AG$  and it is denoted by  $[AG : AH]$ .

(vi) The set of all distinct left cosets of  $AH$  in  $AG$  denoted by  $(AG/AH)_L$  is defined by

$$(AG/AH)_L = \{x * AH : x \in AG\}.$$

(vii) The set of all distinct right cosets of  $AH$  in  $AG$  denoted by  $(AG/AH)_R$  is defined by

$$(AG/AH)_R = \{AH * x : x \in AG\}.$$

Suppose that  $AG/AH$  is the set of all distinct left cosets of  $AH$  in  $AG$  and suppose that  $\odot$  is a binary operation on  $AG/AH$  defined by

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x * AH, y * AH \in AG/AH.$$

If the couple  $(AG/AH, \odot)$  is an AntiGroup of type-AG[4], then  $AG/AH$  is called an AntiQuotientGroup of  $AG$  factored by  $AH$ .

**Lemma 3.15.** Let  $AH$  be an AntiSubgroup of the AntiGroup  $(AG, *)$  of type-AG[4] and let  $e \in AG$  be a NeutroNeutralElement. Then globally,

$$e * AH \neq AH.$$

**Example 3.16.** Let  $(AG, \oplus)$  be an AntiGroup of Example 3.13 (i) and let  $AH = \{0, 1, 2\}$  be its AntiSubgroup. The left and right cosets of  $AH$  in  $AG$  are:

$$\begin{aligned} 0 \oplus AH &= \{0, 1, 2\} \text{ or } \{1, 2\} = AH \oplus 0, \\ 1 \oplus AH &= \{1, 2, 3\} = AH \oplus 1 \\ 2 \oplus AH &= \{0, 2, 3\} \text{ or } \{2, 3\} = AH \oplus 2, \\ 3 \oplus AH &= \{1, 3, ?\} = AH \oplus 3, \\ \therefore AG/AH &= \{0 \oplus AH, 1 \oplus AH, 2 \oplus AH, 3 \oplus AH\}. \end{aligned}$$

It is noted that  $AH$  is a normal AntiSubgroup of  $AG$  and distinct left and right cosets of  $AH$  do not partition  $AG$ .

Now consider the Cayley table below.

$\oplus$	$0 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$0 \oplus AH$	$0 \oplus AH$ or $1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$
$1 \oplus AH$	$1 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $2 \oplus AH$
$2 \oplus AH$	$2 \oplus AH$	$3 \oplus AH$	$0 \oplus AH$ or $3 \oplus AH$	$1 \oplus AH$
$3 \oplus AH$	$3 \oplus AH$	?	$1 \oplus AH$	$2 \oplus AH$

It is evident from the table that  $(AG/AH, \oplus)$  is an AntiGroup of type-AG[4].

**Example 3.17.** Let  $(AG, \otimes)$  be an AntiGroup of Example 3.13 (ii) and let  $AH = \{1, 2, 3\}$  be its AntiSubgroup. The left cosets of  $AH$  in  $AG$  are:

$$\begin{aligned} 1 \otimes AH &= \{?, 2, 3\} = AH \otimes 1, \\ 2 \otimes AH &= \{0, 2, 4\} = AH \otimes 2, \\ 3 \otimes AH &= \{3, 4, ?\} = AH \otimes 3, \\ 4 \otimes AH &= \{2, 3, 4\} = AH \otimes 4, \\ \therefore AG/AH &= \{1 \otimes AH, 2 \otimes AH, 3 \otimes AH, 4 \otimes AH\}. \end{aligned}$$

It is noted that  $AH$  is a normal AntiSubgroup of  $AG$  and distinct left and right cosets of  $AH$  do not partition  $AG$ .

Now consider the Cayley table below.

$\otimes$	$1 \otimes AH$	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$1 \otimes AH$	?	$2 \otimes AH$	$3 \otimes AH$	$4 \otimes AH$
$2 \otimes AH$	$2 \otimes AH$	$4 \otimes AH$	$0 \otimes AH$	$3 \otimes AH$
$3 \otimes AH$	$3 \otimes AH$	?	$4 \otimes AH$	$2 \otimes AH$
$4 \otimes AH$	$4 \otimes AH$	$3 \otimes AH$	$2 \otimes AH$	$0 \otimes AH$

It is evident from the table that  $(AG/AH, \otimes)$  is an AntiGroup of type-AG[4].

**Proposition 3.18.** Let  $AH$  be a normal AntiSubgroup of an AntiGroup  $(AG, *)$  of type-AG[4] and let  $AG/AH$  be the set of distinct left cosets of  $AH$  in  $AG$ . For  $x * AH, y * AH \in AG/AH$  with  $x, y \in AG$ , let  $\odot$  be a binary operation defined on  $AG/AH$  by

$$(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x, y \in AG.$$

Then,  $(AG/AH, \odot)$  is an AntiGroup of type-AG[4].

*Proof.* Suppose that  $AH$  is a normal AntiSubgroup of an AntiGroup  $(AG, *)$  of type-AG[4] and suppose that the composition of elements in  $AG/AH$  is given by  $(x * AH) \odot (y * AH) = (x * y) * AH \quad \forall x, y \in AG$ . Then there exist some duplets  $(x, y), (u, v), (p, q) \in AG$  such that  $x * y \in AG$  (inner-defined) and  $[u * v = \text{indeterminate or } p * q \notin AG]$  (outer-defined/falsehood). Hence,  $\odot$  satisfies the NeutroClosureLaw. Next, there exist some triplets  $(x, y, z), (p, q, r), (u, v, w) \in AG$  such that  $x * (y * z) = (x * y) * z$  (inner-defined) and  $[[p * (q * r)] \text{ or } [(p * q) * r] = \text{indeterminate or } u * (v * w) \neq (u * v) * w]$  (outer-defined/falsehood). This again shows that  $\odot$  satisfies the NeutroAssociativityAxiom. Also, there exists an element  $e \in AG$  such that  $x * e = e * x = x$  (inner-defined) and  $[[x * e] \text{ or } [e * x] = \text{indeterminate or } x * e \neq x \neq e * x]$  (outer-defined/falsehood) for at least one  $x \in AG$ . This shows the existence of NeutroNeutralElement in  $AG$  and hence there exists a NeutroNeutralElement  $e * AH \in AG/AH$ . Again for all  $x \in AG$ , there does not exist  $u \in AG$  such that  $x * u = u * x = e$ . This is an AntiAxiom of existence of inverse element in  $AG$  and consequently, no element  $x * AH \in AG/AH$  has an inverse. Lastly, there exist some duplets  $(x, y), (u, v), (p, q) \in AG$  such that  $x * y = y * x$  (inner-defined) and  $[[u * v] \text{ or } [v * u] = \text{indeterminate or } p * q \neq q * p]$  (outer-defined/falsehood). This shows that  $\odot$  satisfies the NeutroCommutativityAxiom. Hence,  $(AG/AH, \odot)$  is an AntiGroup of type-AG[4].  $\square$

**Definition 3.19.** Let  $(AG, *)$  and  $(AH, \circ)$  be any two AntiGroups of type-AG[4]. The mapping  $\phi : AG \rightarrow AH$  is called an AntiGroupHomomorphism if  $\phi$  does not preserve the binary operations  $*$  and  $\circ$  that is for all the duplet  $(x, y) \in AG$ , we have

$$\phi(x * y) \neq \phi(x) \circ \phi(y).$$

The kernel of  $\phi$  denoted by  $Ker\phi$  is defined by

$$Ker\phi = \{x : \phi(x) = e_{AH} \text{ for at least one } e_{AH} \in AH\}$$

where  $e_{AH}$  is a NeutroNeutralElement in  $AH$ .

The image of  $\phi$  denoted by  $Im\phi$  is defined by

$$Im\phi = \{y \in AH : y = \phi(x) \text{ for some } x \in AG\}.$$

If in addition  $\phi$  is an AntiBijection, then  $\phi$  is called an AntiGroupIsomorphism. AntiGroupEpimorphism, AntiGroupMonomorphism, AntiGroupEndomorphism, and AntiGroupAutomorphism are defined similarly.

**Example 3.20.** (i) Let  $(AG, \oplus)$  be the AntiGroup of Example 3.13 (i) and let  $\phi : AG \rightarrow AG$  be a mapping defined by

$$\phi(x) = 2 \oplus x \quad \forall x \in AG.$$

Then

$$\begin{aligned}\phi(0) &= 2, \\ \phi(1) &= 3, \\ \phi(2) &= 0 \text{ or } 3, \\ \phi(3) &= 1,\end{aligned}$$

from which we obtain that  $\phi(x \oplus y) \neq \phi(x) \oplus y$  for all  $x, y \in AG$ . Accordingly,  $\phi$  is an AntiGroupHomomorphism.  $Im\phi = \{1, 2, 3\}$  which is an AntiSubgroup of  $AG$ .  $Ker\phi = \{\} = \emptyset$ .

- (ii) Let  $(AG, \otimes)$  be the AntiGroup of Example 3.13 (ii) and let  $\psi : AG \rightarrow AG$  be a mapping defined by

$$\psi(x) = x \otimes 4 \quad \forall x \in AG.$$

Then

$$\begin{aligned}\psi(1) &= 4, \\ \psi(2) &= 3, \\ \psi(3) &= 2, \\ \psi(4) &= 0,\end{aligned}$$

from which we obtain that  $\psi(x \otimes y) \neq \psi(x) \otimes y$  for all  $x, y \in AG$ . Accordingly,  $\psi$  is an AntiGroupHomomorphism.  $Im\psi = \{0, 2, 3, 4\}$  which is not an AntiSubgroup of  $AG$ .  $Ker\psi = \{\} = \emptyset$ .

**Example 3.21.** (i) Let  $(AG, *)$  be the AntiGroup of Example 3.10 (i) and let  $\phi : AG \times AG \rightarrow AG$  be a projection defined by

$$\phi((x, y)) = x \quad \forall x, y \in AG.$$

Then  $\phi$  is not an AntiGroupHomomorphism because  $\phi((a, b) \otimes (b, c)) = \phi((a, b)) * \phi((b, c)) = a$ . However,  $Im\phi = \{a, b, c, e\} = AG$ .

- (ii) Let  $(AG, *)$  be the AntiGroup of Example 3.10 (ii) and let  $\psi : AG \times AG \rightarrow AG$  be a projection defined by

$$\psi((x, y)) = y \quad \forall x, y \in AG.$$

Then  $\psi$  is not an AntiGroupHomomorphism because  $\psi((a, b) \otimes (b, c)) = \psi((a, b)) * \psi((b, c)) = c$ . However,  $Im\psi = \{a, b, c, e\} = AG$ .

**Example 3.22.** (i) Let  $(AG/AH, \oplus)$  be the AntiQuotientGroup of Example 3.16 and let  $\phi : AG \rightarrow AG/AH$  be a mapping defined by

$$\phi(x) = x \oplus AH \quad \forall x \in AG.$$

Then

$$\begin{aligned}\phi(0) &= 0 \oplus AH, \\ \phi(1) &= 1 \oplus AH, \\ \phi(2) &= 2 \oplus AH, \\ \phi(3) &= 3 \oplus AH,\end{aligned}$$

from which we obtain

$$\phi(1 \oplus 2) = \phi(1) \bigoplus \phi(2) = 3 \oplus AH.$$

This shows that  $\phi$  is not an AntiGroupHomomorphism.

- (ii) Let  $(AG/AH, \otimes)$  be the AntiQuotientGroup of Example 3.17 and let  $\psi : AG \rightarrow AG/AH$  be a mapping defined by

$$\psi(x) = x \otimes AH \quad \forall x \in AG.$$

Then

$$\begin{aligned}\psi(1) &= 1 \otimes AH, \\ \psi(2) &= 2 \otimes AH, \\ \psi(3) &= 3 \otimes AH, \\ \psi(4) &= 4 \otimes AH,\end{aligned}$$

from which we obtain

$$\psi(2 \otimes 4) = \psi(2) \bigotimes \psi(4) = 3 \otimes AH.$$

This shows that  $\psi$  is not an AntiGroupHomomorphism.

**Remark 3.23.** The fundamental theorem of homomorphisms of the classical groups cannot hold in the class of AntiGroups of type-AG[4] as demonstrated in Examples 3.22 (i) and (ii).

## 4 Conclusion

The notion of AntiGroups was formally presented in this paper. A particular class of AntiGroups of type-AG[4] was studied. In AntiGroups of type-AG[4], the existence of an inverse element was taking to be totally false for all the elements while the closure law, the existence of identity element, the axioms of associativity and commutativity were taking to be either partially true, partially indeterminate or partially false for some elements. It was shown that some algebraic properties of the classical groups do not hold in the class of AntiGroups of type-AG[4]. Specifically, it was shown that intersection of two AntiSubgroups is not necessarily an AntiSubgroup and the union of two AntiSubgroups may be an AntiSubgroup. Also, it was shown that distinct left(right)cosets of AntiSubgroups of AntiGroups of type-AG[4] do not partition the AntiGroups; and that Lagranges' theorem and fundamental theorem of homomorphisms of the classical groups do not hold in the class of AntiGroups of type-AG[4]. More classes of AntiGroups will be studied in our future papers.

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